The Yang-Lee Edge Singularity on Feynman Diagrams

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Abstract

We investigate the Yang-Lee edge singularity on non-planar random graphs, which we consider as the Feynman Diagrams of various $d = 0$ field theories, in order to determine the value of the edge exponent σ.

We consider the hard dimer model on ϕ^3 and ϕ^4 random graphs to test the universality of the exponent with respect to coordination number, and the Ising model in an external field to test its temperature independence. The results here for generic ("thin") random graphs provide an interesting counterpoint to the discussion by Staudacher of these models on planar random graphs [1].

1 Introduction

The work of Yang and Lee [2, 3], later expanded by various other authors [4], on the behaviour of spin models in *complex* external fields has provided an important paradigm for the understanding of the nature of phase transitions. In brief, Yang and Lee observed that the partition function of a system above its critical temperature T_c was non-zero throughout some neighbourhood of the real axis in the complex external field plane. As $T \to T_c+$ the endpoints of loci of zeroes moved in to pinch the real axis, signalling the transition. When such endpoints occur at non-physical (i.e. complex) external field values they can be considered as ordinary critical points with an associated edge critical exponent. The picture was later extended by Fisher to temperature driven transitions by considering the analyticity properties of the free energy in the complex temperature plane [5].

A few equations to flesh this out would perhaps not go amiss. On a finite graph G_n with n vertices the free energy of an Ising-like spin model can be written as

$$F(G_n, \beta, z) = -nh - \ln \prod_{k=1}^{n} (z - z_k(\beta)) \tag{1}$$

where the fugacity $z = \exp(-2h)$, and h is the (possibly complex) external field. The $z_k(\beta)$ are the Yang-Lee zeroes, which in the thermodynamic limit form dense sets on curves in the complex z plane. In the infinite volume limit $n \to \infty$ the free energy per spin is

$$F(G_\infty, \beta, z) = -h - \int_{-\pi}^{\pi} d\theta \rho(\beta, \theta) \ln(z - e^{i\theta}) \tag{2}$$

where $\rho(\beta, \theta)$ is the density of the zeroes, which can be shown to appear on the unit circle in the complex z plane in the Ising case (the Yang-Lee circle theorem). For $T > T_c$ or, if one prefers $\beta < \beta_c$, there is a gap with $\rho(\beta, \theta) = 0$ for $|\theta| < \theta_0$, and at these edge singularities we have

$$\rho(\beta, \theta) \sim (\theta - \theta_0)^\sigma \tag{3}$$

which defines the Yang-Lee edge exponent σ. This also implies $M \sim (\theta - \theta_0)^\sigma$. Various finite size scaling relations relate the Yang-Lee exponent to the other critical exponents [6] and can be used in numerical determinations of critical behaviour [7].

The Yang-Lee circle theorem of [2, 3] guarantees that the roots of the partition function of the Ising model on a fixed graph G_n lie on the unit circle, but it does not guarantee that this should be the case when the partition function is defined by a sum over some class of random graphs for each n

$$Z_n = \sum_{G_n} Z(G_n) \tag{4}$$

where $Z(G_n)$ is the partition function on a given graph in the class. This is the case for models of 2D quantum gravity, where the sum is over planar $\phi^3, \phi^4 \ldots$ random graphs, and in this paper where we will will consider a sum over thin, *non*-planar random graphs [1]. The work of Staudacher [1] showed that, nonetheless, the zeroes did appear on the unit circle for planar graphs, which has recently been confirmed by numerical finite-size scaling investigations using both series expansions and Monte-Carlo simulations by Ambjørn et.al. [8].

In this paper we will consider a partition function of the form equ.(4) for thin random graphs. Spin models on such thin random graphs are of interest because they give a way of investigating mean-field effects thanks to their tree-like local structure [9]. The advantage of using the thin random graphs in such investigations over genuine tree-like structures such as the Bethe lattice [10] is that boundary effects are absent. The complications of being forced to consider only sites deep within the lattice which occur on the Bethe lattice are thus absent. The motivation for the current work is the calculation of the exponent

[1]We shall call such graphs "thin" random graphs throughout this paper as they appear as the scalar limit of the matrix fatgraphs that are relevant for discussions of 2D gravity.

σ for thin random graphs and the testing of its universality. The calculation of the edge exponent in the Ising model context also allows us to check its temperature independence. A mean-field value for σ (i.e. 1/2) would demonstrate that the mean-field nature of critical behaviour on such random graphs models extended to complex couplings.

We shall use the approach of [12, 13] to solve both the hard dimer model and the Ising model itself in a complex external field. The requisite ensemble of thin random graphs is generated by considering the scalar limit of a matrix model. In all cases the partition function of equ.(4) is given by an integral of the form

$$Z_n \times N_n = \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda^{2n+1}} \int \frac{\prod_i d\phi_i}{2\pi\sqrt{\det K}} \exp(-S), \tag{5}$$

where K is the inverse of the quadratic part of the action S, the ϕ_i are the fields which give the appropriate decoration of the graph, and λ is the vertex coupling. The factor N_n counts the number of undecorated graphs in the class of interest, and generically grows factorially with n. A given graph appears as a particular "Feynman diagram" in the expansion of equ.(5) and the integration over λ picks out graphs with $2n$ vertices. The coupling λ is irrelevant for the discussion of critical behaviour as it may be scaled out of the action and hence any saddle point equations. In the large n limit the integral in equ.(5) may be evaluated by saddle point methods. Phase transitions appear when an exchange of dominant saddle points occurs, either continuously giving a second order transition, or discontinuously giving a first order transition. The saddle point integrals which appear are the $d = 0$ equivalent of those in instanton and large orders calculations in field theory [14, 15, 16]. As we are taking an $n \to \infty$ limit at the start of our calculations, we are unable to explicitly calculate zeroes on finite lattices and verify the Yang-Lee circle theorem. However, we shall see that the end-points of the loci of zeroes do lie on the unit circle in the complex fugacity plane.

One important property of the Yang-Lee edge exponent σ in all the models examined so far is that it is independent of β (for $\beta < \beta_c$). It is therefore possible in general to get a quick and dirty determination of its value for Ising models by taking the so-called hard dimer limit, which corresponds to $\beta \to 0, h \to \frac{i\pi}{2}$, and has certain simplifying features compared with the general case. We will calculate the edge exponent for the hard dimer model on ϕ^3 and ϕ^4 random graphs in the next section. A calculation of the exponent for the Ising model proper in the section which follows this will investigate whether the temperature independence still holds.

2 Hard Dimer Models

The partition function of the hard dimer model on a given graph is defined [17] by

$$\Theta(G_n) = 1 + \sum_{i=1}^{e(G_n)} \theta_n(i)\zeta^i \tag{6}$$

where ζ is a dimer activity and $\theta_n(i)$ is the number of ways of placing i dimers on the $e(G_n)$ edges of the graph G_n such that at most one dimer is attached to each vertex (hence "hard"). Precisely this expression appears in taking the limit $\beta \to 0, h \to \frac{i\pi}{2}$ in the high temperature series for the Ising model partition function. The role of h in the definition of the edge singularity exponent is taken by ζ and we have

$$\frac{d\Theta}{d\zeta} \sim (\zeta - \zeta_0)^\sigma \tag{7}$$

where Θ is the appropriate $n \to \infty$ limit of $\Theta(G_n)$. For both planar and thin random graphs we are interested in a sum

$$\Theta_n = \sum_{G_n} \Theta(G_n) \tag{8}$$

and in [1] the appropriate two matrix integral to generate the dimer partition function on ϕ^3 and ϕ^4 planar graphs was written down. For thin ϕ^3 graphs the required action for insertion into equ.(4) is

$$S = \frac{1}{2}\left(x^2 + y^2\right) - \frac{1}{3}x^3 - \sqrt{\zeta}yx^2. \tag{9}$$

where the x propagators represent the unoccupied links and the y propagators the dimers. The two vertices are shown in Fig.1. The $\sqrt{\zeta}$ weight appears because each end of the dimer contributes a $\sqrt{\zeta}yx^2$ vertex. We have scaled out λ for clarity. Similarly for ϕ^4 graphs we find

$$S = \frac{1}{2}\left(x^2 + y^2\right) - \frac{1}{4}x^4 - \sqrt{\zeta}yx^3. \tag{10}$$

The saddle point equations $\partial S/\partial x = \partial S/\partial y = 0$ for both equs.(9,10) can be easily solved. Concentrating on the ϕ^3 case for simplicity, we find

$$x = -\frac{1 \pm \sqrt{1+8\zeta}}{4\zeta}, \quad y = \frac{1-x}{2\sqrt{\zeta}} \tag{11}$$

so the resulting saddle point action is

$$S = \frac{1}{192\zeta^3}\left(12\zeta - 8\zeta\sqrt{1+8\zeta} + 24\zeta^2 + 1 - \sqrt{1+8\zeta}\right). \tag{12}$$

We can see that the saddle point solution presents a singularity at a negative value, $\zeta_0 = -1/8$. The free energy is given by the logarithm of the action to leading order in $1/n$, so we would expect an inverse square root divergence at ζ_0 when we differentiate $\ln S$ twice if σ were equal to its mean-field value of $1/2$. This is, indeed, the case. Writing $\zeta = -1/8 + \epsilon$ and expanding we find

$$\frac{\partial^2\Theta}{\partial\zeta^2} \sim \frac{\partial^2 \ln S}{\partial\zeta^2} \sim \frac{96\sqrt{2}}{\sqrt{\epsilon}}. \tag{13}$$

As y appears at most quadratically in equ.(9) an alternative approach [2] is to integrate it out to get the action

$$S = \frac{1}{2}\left(x^2 + y^2\right) - \frac{1}{3}x^3 - \frac{1}{2}\zeta x^4 \tag{14}$$

which has the saddle point solution

$$x = -\frac{1 \pm \sqrt{1+8\zeta}}{4\zeta} \tag{15}$$

and displays an identical divergence $\sim 96\sqrt{2}/\sqrt{\epsilon}$ at $\zeta_0 = -1/8$ to equ.(13). The geometrical picture of such an integration on the y variables is that all the dimers are collapsed to give new x^4 vertices and assigned the appropriate weight. Whichever way the calculation is carried out, the appearance of the square root divergence confirms that $\sigma = 1/2$, which is the mean-field value of the edge exponent.

The universality of the result with respect to the coordination number of the vertices can be confirmed by solving the saddle point equations for equ.(10), which we do not reproduce here, or by integrating out y to give

$$S = \frac{1}{2}\left(x^2 + y^2\right) - \frac{1}{4}x^4 - \frac{1}{2}\zeta x^6 \tag{16}$$

which has the saddle point solution

$$x = \frac{\sqrt{-1 \pm \sqrt{1+8\zeta}}}{\sqrt{6}\zeta} \tag{17}$$

leading to the saddle point action

$$S = \frac{1}{432}\frac{(-1+\sqrt{1+12\zeta})(24\zeta + 1 - \sqrt{1+12\zeta})}{\zeta^2}. \tag{18}$$

When expanded around the singularity at $\zeta_0 = -1/12$ it also gives a square root divergence

$$\frac{\partial^2\Theta}{\partial\zeta^2} \sim \frac{\partial^2 \ln S}{\partial\zeta^2} \sim \frac{36\sqrt{3}}{\sqrt{\epsilon}}. \tag{19}$$

[2] Also taken in [1] for the planar case in a matrix model calculation.

The exponent σ is thus seen to be independent of the coordination number of the random graphs (three and four for ϕ^3 and ϕ^4 graphs, respectively) on which the dimers are placed.

The hard dimer calculation represents a determination of the edge exponent at one point on the (h,β) plane. We now turn to the solution of the Ising model in an external field in order to confirm this value for generic points on the singular $h(\beta)$ line.

3 The Ising Model in an External Field

The action for the Ising model on ϕ^3 graphs in an external field may be written as

$$S = \frac{1}{2}\left(x^2+y^2\right) - cxy - \frac{1}{3}e^h x^3 - \frac{1}{3}e^{-h}y^3 \tag{20}$$

where $c = \exp(-2\beta)$. The transition point in the model is determined by first solving the saddle point equations $\partial S/\partial x = \partial S/\partial y = 0$ and then using these solutions to determine at which point the Hessian of the second partial derivatives is zero [11]. This will pick up any continuous transitions that are present. The net result of these (lengthy) calculations is the following formula for $h(c)$, the curve in the h,c plane along which the Hessian is zero

$$\exp(h(c)) = \pm\frac{\sqrt{2c\left(1+18c^2-27c^4 \pm (1-9c^2)\sqrt{1-10c^2+9c^4}\ \right)}}{4c} \tag{21}$$

It is perhaps worthwhile to consider the solution in zero external field at this point for orientational purposes. In that case we have at high temperatures

$$x = y = 1-c \tag{22}$$

which bifurcates in a mean-field magnetisation transition at $c = 1/3$ to the low temperature solutions

$$\begin{aligned} x &= \frac{1+c+\sqrt{1-2c-3c^2}}{2} \\ y &= \frac{1+c-\sqrt{1-2c-3c^2}}{2}. \end{aligned} \tag{23}$$

valid for $c < 1/3$. The distinguished role of the zero-field critical point $c = 1/3$ is clear in equ.(21). $\exp(h(c))$ develops an imaginary part for $c > 1/3$ (in the high temperature phase) and if we plot its modulus as in Fig.2 we can see clearly that $|\exp(h(c))| = 1$ for $c > 1/3$. This shows that the endpoints, at least, of the line of zeroes lie on the unit circle. Without explicit finite size calculations, or simulations of the model in a complex field in the style of [8] we cannot say that the zeroes lie on the unit circle, but it is reasonable to conjecture that they do, just as for planar graphs.

Extracting the critical exponent from the magnetisation turns out to be easiest proposition for the Ising model in a field. The expression for the magnetisation in this case is

$$M(e(h)) = \frac{e^h x^3 - e^{-h}y^3}{e^h x^3 + e^{-h}y^3} \tag{24}$$

and it is directly related to the density of zeroes on the unit circle by

$$\rho(\theta) \sim \lim_{\to 1-} Re\, M\, \left(re^{i\theta}\right). \tag{25}$$

We proceed by substituting the saddle point solutions for x, y in field that led to equ.(21) into equ.(24) before taking the limit in equ.(25) above. The positions of the critical endpoints on the unit circle can be extracted by examining the discontinuities in the resulting expression, and are given by

$$\theta_0(c) = \pm\frac{1}{2}\tan^{-1}\left(\frac{(9c^2-1)\sqrt{1-10c^2+9c^4}}{1+18c^2-27c^4}\right) \tag{26}$$

which is also consistent with equ.(21) for $\exp(h)$. The endpoints can be seen to move in to pinch the real axis, $\theta_0 \to 0\pm$ as $c \to 1/3+$, confirming the Yang-Lee picture of the transition.

An expansion of the density of zeroes around θ_0 for generic c still rapidly degenerates into considerable, and not very illuminating, algebraic complexity. However, fixing c and examining various points along the critical line in equ.(21) reduces the level of difficulty to that of the hard dimer calculations in the previous section. For any given value of $c > 1/3$, we do indeed find that $\rho(\theta) \sim (\theta - \theta_0(c))^{1/2}$ by using equ.(25). In Fig.3 the density of zeroes is plotted against $\theta - \theta_0(c)$ for $c = 1/2$ and the square root nature of the singularity is evident, as can be confirmed by fitting to the curve for this and other values of c.

4 Discussion

Both the hard dimer calculations and those for the full Ising model in an external field produce the mean-field value for the edge exponent. Given the body of previous results in [11] showing mean-field behaviour in various models on random graphs (and the observation in the first of [11] that the saddle point equations in the mean-field models were identical in content to the recursion relations use to solve the models on trees) this is no great surprise, though it does provide the first confirmation that the mean-field critical behaviour on random graphs extends to critical phenomena at complex couplings. The inherent simplicity of the saddle point equations for random graphs made obtaining expressions for the singular behaviour and the position of the critical endpoints a simple task in the hard dimer model and also allowed for a demonstration of universality by examining both ϕ^3 and ϕ^4 graphs. The solution of the Ising model in field was rather more forbidding, and we have not presented many of the resulting elephantine equations here, but it was still possible to give a simple demonstration by semi-numerical means that $\sigma = 1/2$ along the critical curve.

We have also seen that for the Ising model the singular endpoints of the line of zeroes stay on the unit circle in the complex fugacity plane, which provides support for the Yang-Lee circle theorem with partition functions of the form in equ.(4) that involve sums over thin random graphs. This naturally leads to the conjecture that all the Yang-Lee zeroes lie on the unit circle for the Ising model on thin random graphs, just as on planar graphs, which could be confirmed by a finite size scaling analysis for the thin random graph model in the manner of that carried out in [8] for planar graphs. Another possible extension of the present work would be to consider the Fisher zeroes in the model by examining the behaviour in the complex temperature plane.

Indeed, a more comprehensive investigation of complex phases for various spin and vertex models on both planar and non-planar random graphs, in the manner of that carried out by Matveev and Shrock [18] for regular lattices, might prove illuminating. In particular, it would be interesting to see if the complex temperature and complex field singularities with atypical (and lattice dependent) exponents found in [18] had their counterparts in random graph models.

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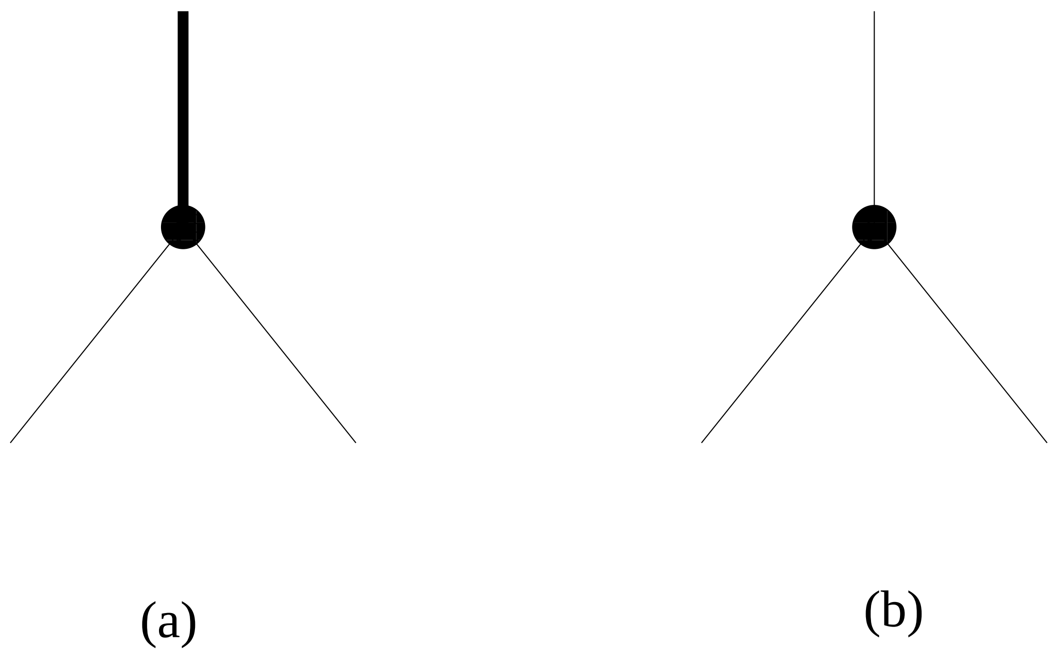

Figure 1: The two vertices in the ϕ^3 dimer model, with the dimer edge drawn in bold. (a) carries a weight of $\lambda\sqrt{\zeta}$ and (b) carries a weight of $\lambda/3$.

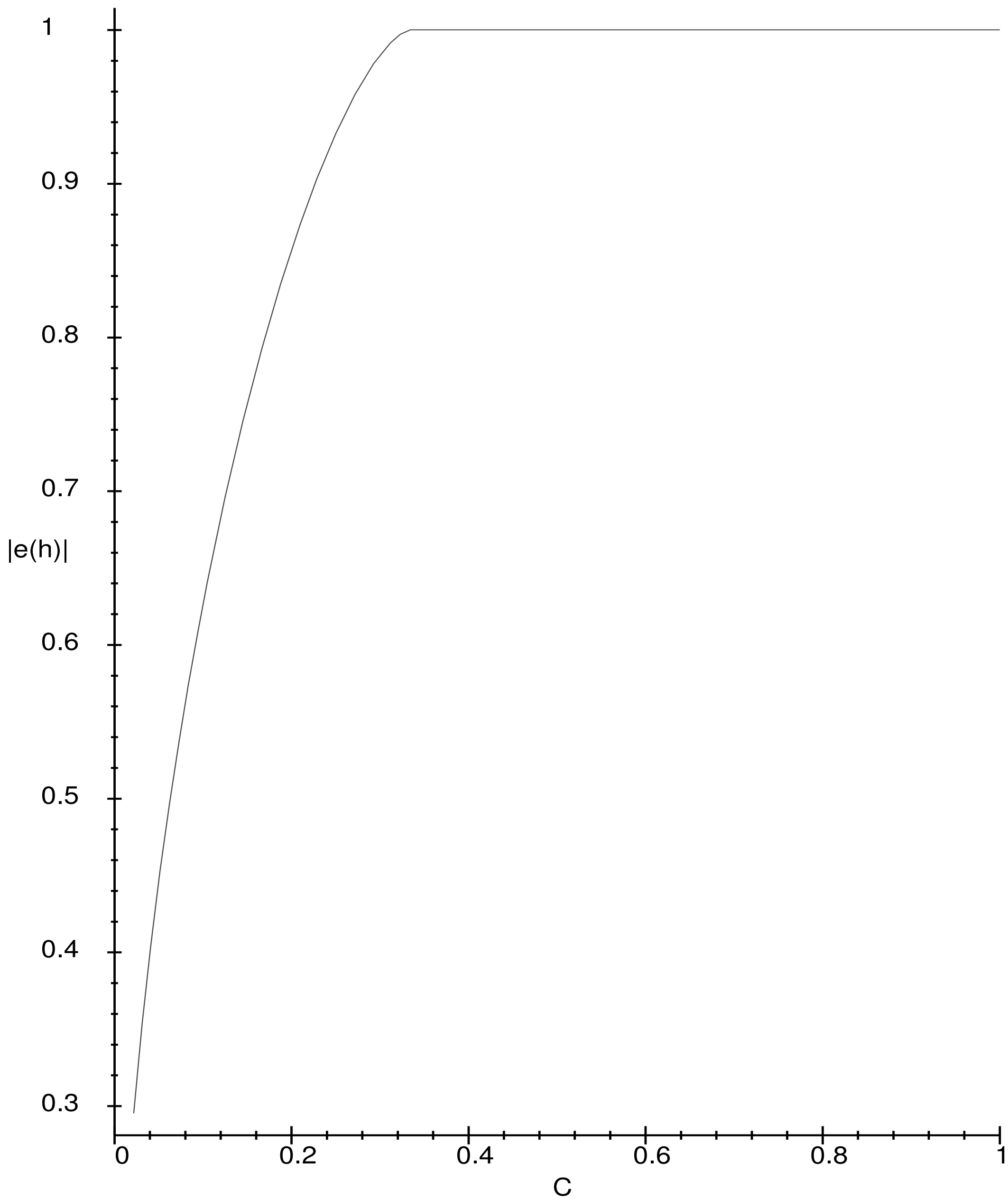

Figure 2: The modulus of exp(h) along the Yang-Lee transition line for $c > 1/3$ can be seen to be one

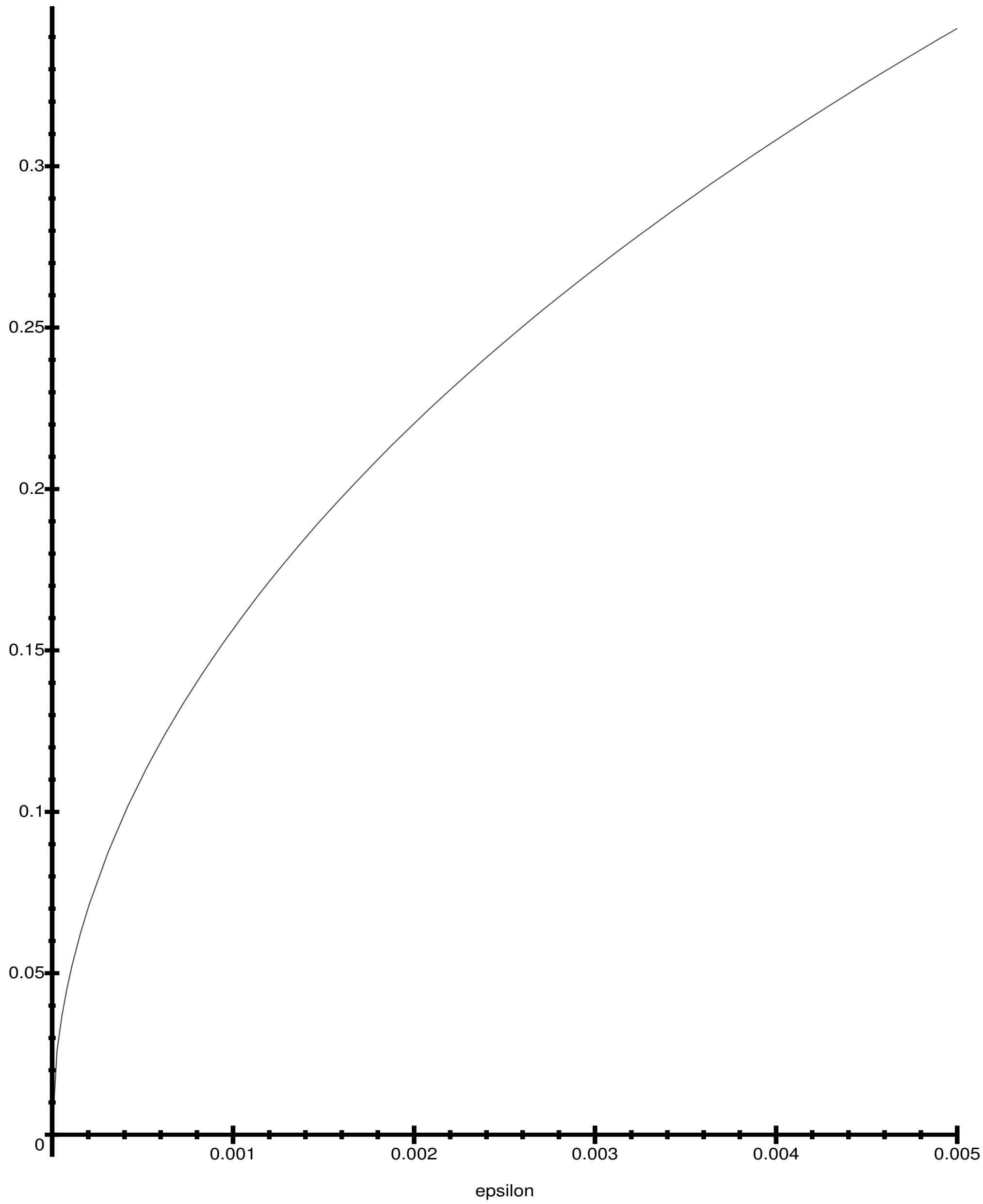

Figure 3: The density of zeroes ρ (suitably scaled for plotting convenience) is plotted against $\epsilon = \theta - \theta(c)$ for $c = 1/2$ to show the square root edge singularity.